Math 10A
Worksheet, Discussion #6; Monday, 6/25/2018
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1 Derivatives and Tangents

1.1 Concepts

1. The point slope formula for a line with slope m going through the point (x_0, y_0) is
$$y - y_0 = m(x - x_0).$$
The derivative of $f^{-1}(x)$ at the point (x, y) is $\frac{1}{f'(y)}$.

1.2 Basic Derivatives

2. Find the tangent line to $-x^2$ at $x = 1$.

Solution: Using the point slope formula with $f(x) = -x^2$ gives
$$y - f(1) = f'(1)(x - 1) \implies y + 1 = -2(x - 1) \implies y = -2x + 1.$$

3. Find the derivative of $(\tan x)^2$.

Solution:
$$\frac{d}{dx}(\tan x)^2 = 2\tan x \cdot \frac{d}{dx}\tan x = 2\tan x \sec^2 x.$$

4. Find the derivative of $\frac{x}{1 - \sin x}$.

Solution: This is actually an application of the quotient rule:
$$\frac{d}{dx}\frac{x}{1 - \sin x} = \frac{(1 - \sin x) \cdot \frac{d}{dx}x - x\frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} = \frac{1 - \sin x - x(-\cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x + x\cos x}{(1 - \sin x)^2}.$$

5. Find the tangent line to x^3 at $x = -1$.

Solution: We use the point slope formula with $f(x) = x^3$ to get

$$y - f(-1) = f'(-1)(x - (-1)) \implies y - (-1) = 3(-1)^2(x+1) \implies y = 3x + 2.$$

6. Find the derivative of $e^{\sin(2x)}$.

Solution:

$$\frac{d}{dx} e^{\sin(2x)} = e^{\sin(2x)} \cdot \frac{d}{dx} \sin(2x) = e^{\sin(2x)} \cdot \cos(2x) \cdot \frac{d}{dx}(2x) = 2\cos(2x) e^{\sin(2x)}.$$

1.3 Inverse Derivatives

7. Let $f(x) = x^5 + 3x^3 + 7x + 2$. Find the tangent line to $f^{-1}(x)$ at $(13, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx} f^{-1}(13)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(13) = \frac{1}{f'(f^{-1}(13))} = \frac{1}{f'(1)} = \frac{1}{21}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{x - 13}{21} \implies y = \frac{x+8}{21}.$$

8. Let $f(x) = e^{-2x} - 9x^3 + 4$. Find the tangent line to $f^{-1}(x)$ at $(5, 0)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx} f^{-1}(5)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(0)} = \frac{-1}{2}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 0 = \frac{x - 5}{-22} \implies y = \frac{-x+5}{2}.$$

9. Let $f(x) = x^7 + 2x + 9$. Find the tangent line to $f^{-1}(x)$ at $(12, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(12)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(12) = \frac{1}{f'(f^{-1}(12))} = \frac{1}{f'(1)} = \frac{1}{9}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{x - 12}{9} \implies y = \frac{x}{9} - \frac{1}{3}.$$

10. Let $f(x) = x^{5/3}e^{x^2}$. Find the tangent line to $f^{-1}(x)$ at $(e, 1)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(e)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(e) = \frac{1}{f'(f^{-1}(e))} = \frac{1}{f'(1)} = \frac{3}{11e}.$$

We found this by using the product rule to find the derivative of f since

$$f'(x) = \frac{5}{3}x^{2/3}e^{x^2} + x^{5/3} \cdot e^{x^2} \cdot 2x.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 1 = \frac{3(x - e)}{11e} \implies y = \frac{3x}{11e} + \frac{8}{11}.$$

11. Let $f(x) = \dfrac{-e^{-3x}}{x^2 + 1}$. Find the tangent line to $f^{-1}(x)$ at $(-1, 0)$.

Solution: In order to find the tangent, we need to know $\frac{d}{dx}f^{-1}(-1)$ and we can find this using the formula:

$$\frac{df^{-1}}{dx}(-1) = \frac{1}{f'(f^{-1}(-1))} = \frac{1}{f'(0)} = \frac{1}{3}.$$

We found this by using the quotient rule to find the derivative of f since

$$f'(x) = \frac{(x^2 + 1) \cdot (-e^{-3x}) \cdot (-3) - (-e^{-3x}) \cdot (2x)}{(1 + x^2)^2}.$$

Now we plug that into the point slope formula to get the line

$$y - y_0 = m(x - x_0) \implies y - 0 = \frac{x - (-1)}{3} \implies y = \frac{x+1}{3}.$$

1.4 Implicit Derivatives

12. Find y' if $x^3 + y^3 = 4$.

Solution: Taking the derivative of both sides, we have that $3x^2 + 3y^2 \cdot y' = 0$ and hence $y' = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$.

13. Find y' if $e^{xy} = e^{4x} - e^{5y}$.

Solution: We take the derivative to get that $4e^{4x} - 5e^{5y}y' = e^{xy}(xy)' = e^{xy}(xy' + y)$. Bringing all the y' to one side, we have $4e^{4x} - ye^{xy} = 5e^{5y}y' + xe^{xy}y'$ so $y' = \dfrac{4e^{4x} - ye^{xy}}{5e^{5y} + xe^{xy}}$.

14. Find y' if $(x-y)^2 = x + y - 1$.

Solution: Taking the derivative gives $2(x-y)(1-y') = 1 + y'$ so $2x - 2y - 1 = y'(1 + 2x - 2y)$ so $y' = \dfrac{2x - 2y - 1}{2x - 2y + 1}$.

15. Find y' if $y = \sin(3x + 4y)$.

Solution: Taking the derivative gives $y' = \cos(3x+4y)(3x+4y)' = \cos(3x+4y)(3+4y')$. Thus, $y' = \dfrac{3\cos(3x+4y)}{1 - 4\cos(3x+4y)}$.

16. Find y' if $y = x^2y^3 + x^3y^2$.

Solution: Taking the derivative gives $y' = 2xy^3 + 3x^2y^2y' + 3x^2y^2 + 2x^3yy'$ and hence $y' = \dfrac{2xy^2 + 3x^2y^2}{1 - 3x^2y^2 - 2x^3y}$.